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On Mathematical Aspects of Dual Variables in Continuum Mechanics. Part 2: Applications in Nonlinear Solid Mechanics

In Fortsetzung des Vorgehens von Teil 1 wenden wir unsere Ergebnisse auf duale Variable an, die in der Kontinuumsmechanik auftreten. Dazu untersuchen wir die Kinematik und Dynamik kontinuierlicher Körper. Es folgt eine gemischtvariante Formulierung der Kinematik und der Cauchyschen Beziehung für den Spannungstensor. Die gemischtvariante Beschreibung der Kinematik bietet Vorteile in der finiten Plastizitätstheorie. Schließlich werden noch Beispiele für Stoffgleichungen angegeben, die in exemplarischer Weise die zusätzliche mathematische Struktur aufdecken, die durch das vorgeschlagene Vorgehen gewonnen wird.

Continuing the approach of Part 1 of this paper we apply our results to dual variables appearing in continuum mechanics. We investigate the kinematics and dynamics of continuous bodies. As a consequence, this leads to a mixed variant formulation of kinematics and Cauchy's law for the stress tensor. The mixed variant formulation of kinematics is advantageous, for instance, in finite deformation plasticity. Finally, we give some examples of constitutive equations which demonstrate in an exemplaric manner the additional mathematical structure gained through our approach.

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In Part 2 of our paper we first give a very brief summary of the mathematical theory presented in Part 1. Then we will apply our results to the mechanics of nonlinear solids.

5. Summary of Part 1

In Part 1 of this paper we have distinguished between scalar and inner products. Scalar products are formed from objects living in dual spaces while inner products are formed from objects living in one and the same space. The notion of scalar products leads to the definition of dual or adjoint tensors. Inner products lead in a natural way to transposed tensors. The algebra of such tensors on either the tangent or cotangent spaces of manifolds has been elaborated.

Of great importance for nonlinear continuum mechanics are diffeomorphic maps between manifolds. An essential role is played by the tangent of such a map, which is a mixed two-point tensor. Invariance requirements of certain quadratic forms lead to formulae for the computation of push-forwards and pull-backs. Invariance of inner products yields the transformation rules for mixed tensors. Invariance of scalar products leads to push-forwards and pull-backs, respectively, of covariant and contravariant tensors. Contrary to earlier results [1] symmetry of mixed tensors is preserved with our formalism.

6. Kinematics of motions of continuous bodies

6.1 Basic kinematic concepts

We consider a body B with a reference configuration $\mathscr{B}$ being a differentiable n-dimensional manifold ($n = 1$ or 2 or 3). Let $\{X^A\}$ denote a coordinate system on a subset $\mathscr{U}$ of $\mathscr{B}$. Furthermore, we define a time scale t with $t \in [-\infty, \infty]$. Assume that there exists a one-parameter family $\varphi(t)$ of at least C^1 diffeomorphisms $\varphi(t) : \mathscr{B} \to \mathscr{S}_t$, where the parameter $t \in \mathbb{R}$ is the time. Then $\mathscr{S}_t$ is the *actual configuration* at time t. The actual configuration $\mathscr{S}_t$ is again a manifold. Denote a typical coordinate system on $\mathscr{S}_t$ by $\{x^a\}$. The set of all configurations $\mathscr{S}_t$ is called the *configuration space* $\mathscr{C}$. A motion of the body B is a curve in the configuration space $\mathscr{C}$.

Remark 6.1: In the sequel we will write $\mathscr{S}$ instead of $\mathscr{S}_t$ if no confusion is possible. □

Definition 6.1: The vector field

$$\vec{\mathbf{V}}_t(X) := \frac{\partial\varphi(X, t)}{\partial t} = \frac{\mathrm{d}}{\mathrm{d}t}\,\varphi_X(t) \tag{137}$$

is called the *material velocity* of the motion. The notation $\varphi_X(t)$ indicates that the point X is held fixed. The *material velocity field* is a map $\vec{\mathbf{V}}_t(X) : \mathscr{B} \to T_X\mathscr{B}$. □

Definition 6.2: A motion $\varphi(X, t)$ is called C^r *regular*, if it is a C^r diffeomorphism for all $t \in \mathbb{R}$.

If $\varphi(X, t)$ is an at least C^1 regular motion, then we can define the spatial velocity field.

Definition 6.3: The *spatial velocity* field of an at least C^1 regular motion $\varphi(X, t)$ is defined as

$$\vec{\mathbf{v}}_t := \vec{\mathbf{V}}_t \circ \varphi_t^{-1}\,. \tag{138}$$

The spatial velocity field is a map $\vec{\mathbf{v}} : \mathscr{S} \to T_x\mathscr{S}$. □

Next, we introduce the spatial velocity gradient.

Definition 6.4: Let $\vec{\mathrm{v}}_t$ be a spatial velocity field. Then the *spatial velocity gradient* is defined as

$$\mathbf{l}^{\backslash} := \operatorname{grad} \vec{\mathrm{v}}_t \in T_x\mathscr{S} \otimes T^*_x\mathscr{S}\,, \tag{139}$$

where the notation "grad" indicates that the gradient operation is taken with respect to the current configuration. □

A crucial notion for the description of the kinematics of continuous bodies is the deformation gradient.

Definition 6.5: The *deformation gradient* associated with the motion $\varphi(X, t) : \mathscr{B} \to \mathscr{S}_t$ with t fixed is defined as the tangent of the map $\varphi(X, t)$ introduced in Definition 3.5,

$$\mathscr{F} := T_x\varphi\,. \quad \square \tag{140}$$

Finally we give a well known relation between the velocity gradient and the deformation gradient [14]:

$$\mathbf{l}^{\backslash} = \dot{\mathscr{F}}\mathscr{F}^{-1}\,. \tag{141}$$

6.2 Deformation and strain measures

We compute the squares of the line elements in the reference configuration and in the current configuration. Since $\mathrm{d}\vec{\mathbf{X}} \in T_X\mathscr{B}$ and $\mathrm{d}\vec{\mathrm{x}} \in T_X\mathscr{S}$ are vectors we obtain the squares of the line elements as inner products in $T_X\mathscr{B}$ and $T_x\mathscr{S}$, respectively,

$$\mathrm{d}S^2 = \mathrm{d}\vec{\mathbf{X}} \cdot \mathrm{d}\vec{\mathbf{X}}\,, \qquad \mathrm{d}s^2 = \mathrm{d}\vec{\mathrm{x}} \cdot \mathrm{d}\vec{\mathrm{x}}\,. \tag{142), (143}$$

Since the deformation gradient is the tangent of the map $\varphi(X)$ we can write

$$\mathrm{d}\vec{\mathbf{X}} = \mathscr{F}^{-1}\,\mathrm{d}\vec{\mathrm{x}}\,, \qquad \mathrm{d}\vec{\mathrm{x}} = \mathscr{F}\,\mathrm{d}\vec{\mathbf{X}}\,, \tag{144), (145}$$

where $\mathscr{F}^{-1} : T_x\mathscr{S} \to T_X\mathscr{B}$ is the inverse of the deformation gradient. Applying the rules for transposes of mixed two-point tensors given in Definition 3.3 we obtain from (142) to (145)

$$\mathrm{d}S^2 = \mathrm{d}\vec{\mathrm{x}} \cdot \mathscr{F}^{-\mathsf{T}}\mathscr{F}^{-1}\,\mathrm{d}\vec{\mathrm{x}}\,, \qquad \mathrm{d}s^2 = \mathrm{d}\vec{\mathbf{X}} \cdot \mathscr{F}^{\mathsf{T}}\mathscr{F}\,\mathrm{d}\vec{\mathbf{X}}\,. \tag{146), (147}$$

We point out that (146) and (147) are only possible with the definition of transposition given in Definition 3.3. Only with this definition inner products can be formed.

We define the following deformation tensors,

$$\mathbf{C}^{\backslash} : T_X\mathscr{B} \to T_X\mathscr{B}: \qquad \mathbf{C}^{\backslash} = \mathscr{F}^{\mathsf{T}}\mathscr{F}, \tag{148}$$

$$\mathbf{b}^{\backslash} : T_x\mathscr{S} \to T_x\mathscr{S}: \qquad \mathbf{b}^{\backslash} = \mathscr{F}\mathscr{F}^{\mathsf{T}}, \tag{149}$$

$$\mathbf{c}^{\backslash} : T_x\mathscr{S} \to T_x\mathscr{S}: \qquad \mathbf{c}^{\backslash} = \mathbf{b}^{\backslash -1} = \mathscr{F}^{-\mathsf{T}}\mathscr{F}^{-1}\,. \tag{150}$$

The tensor $\mathbf{C}^{\backslash}$ is called the *right Cauchy-Green tensor* and the tensor $\mathbf{b}^{\backslash}$ is called the *left Cauchy-Green* or *Finger tensor*. From (146), (147), (148), and (150) it now follows that

$$\mathrm{d}S^2 = \mathrm{d}\vec{\mathrm{x}} \cdot \mathbf{c}^{\backslash}\,\mathrm{d}\vec{\mathrm{x}}\,, \qquad \mathrm{d}s^2 = \mathrm{d}\vec{\mathbf{X}} \cdot \mathbf{C}^{\backslash}\,\mathrm{d}\vec{\mathbf{X}}\,. \tag{151), (152}$$

Define the following tensors,

$$\mathbf{E}^{\backslash} := \tfrac{1}{2}\,(\mathbf{C}^{\backslash} - \mathbf{I}^{\backslash}) \in T_X\mathscr{B} \otimes T^*_X\mathscr{B}\,, \tag{153}$$

$$\mathbf{e}^{\backslash} := \tfrac{1}{2}\,(\mathbf{i}^{\backslash} - \mathbf{c}^{\backslash}) \in T_x\mathscr{S} \otimes T^*_x\mathscr{S}, \tag{154}$$

where $\mathbf{I}^{\backslash}$ and $\mathbf{i}^{\backslash}$ are the *mixed identity tensors* on $T_X\mathscr{B} \otimes T^*_X\mathscr{B}$ and $T_x\mathscr{S} \otimes T^*_x\mathscr{S}$, respectively. The tensor $\mathbf{E}^{\backslash}$ is *Green's strain tensor* and the tensor $\mathbf{e}^{\backslash}$ is *Almansi's strain tensor*. With the aid of Table 1 it is straightforward to show that

$$\mathbf{e}^{\backslash} = \varphi_*(\mathbf{E}^{\backslash})\,, \qquad \mathbf{E}^{\backslash} = \varphi^*(\mathbf{e}^{\backslash})\,. \tag{155}$$

From (151) to (154) we obtain the following representations for the differences of the squared line elements in the current and reference configuration, respectively,

$$\mathrm{d}s^2 - \mathrm{d}S^2 = 2\,\mathrm{d}\vec{\mathbf{X}} \cdot \mathbf{E}^{\backslash}\,\mathrm{d}\vec{\mathbf{X}}\,, \qquad = 2\,\mathrm{d}\vec{\mathrm{x}} \cdot \mathbf{e}^{\backslash}\,\mathrm{d}\vec{\mathrm{x}}\,. \tag{156), (157}$$

Remark 6.2.: It is important to note that both the Green and the Almansi strain tensor are essentially mixed tensors (see (153) and (154)). This is a direct consequence of the approach here to distinguish between inner and scalar products. In former investigations, e.g. [1, 13], strain tensors emerge typically as covariant tensors. □

We conclude this section with an example from the kinematics of finitely deformed elasto-plastic solids. All kinematic considerations start with the famous multiplicative decomposition of the deformation gradient [5, 15, 16],

$$\mathscr{F} = \hat{\mathscr{F}}_{\mathrm{e}} \mathscr{F}_{\mathrm{p}} . \tag{158}$$

We call $\hat{\mathscr{F}}_{\mathrm{e}}$ the *elastic part* of the deformation gradient and $\mathscr{F}_{\mathrm{p}}$ its *plastic part*, since for general deformations neither $\hat{\mathscr{F}}_{\mathrm{e}}$ nor $\mathscr{F}_{\mathrm{p}}$ are gradients. It is well known that (158) induces an intermediate configuration $\hat{\mathscr{B}}$ which is obtained from the current configuration $\mathscr{S}$ by relaxing each material particle to a stress free state. Unless the deformation $\varphi : \mathscr{B} \to \mathscr{S}$ is homogeneous, the intermediate configuration is incompatible. Pulling back Almansi's strain tensor from the current configuration to the intermediate configuration we obtain the strain tensor $\hat{\mathbf{E}}^{\backslash}$ that measures the squared difference of line elements through

$$\mathrm{d}s^2 - \mathrm{d}S^2 = 2\, \mathrm{d}\vec{\hat{\mathbf{X}}} \cdot \hat{\mathbf{E}}^{\backslash}\, \mathrm{d}\vec{\hat{\mathbf{X}}} , \tag{159}$$

where

$$\mathrm{d}\vec{\hat{\mathbf{X}}} = \mathscr{F}_{\mathrm{p}}\, \mathrm{d}\vec{\mathbf{X}} = \mathscr{F}_{\mathrm{e}}^{-1}\, \mathrm{d}\vec{\mathbf{x}} \tag{160}$$

is the line element on the intermediate configuration. The strain tensor $\hat{\mathbf{E}}^{\backslash}$ on the intermediate configuration can be decomposed additively as

$$\hat{\mathbf{E}}^{\backslash} = \hat{\mathbf{E}}^{\backslash}_{\mathrm{e}} + \hat{\mathbf{E}}^{\backslash}_{\mathrm{p}} , \tag{161}$$

$$\hat{\mathbf{E}}^{\backslash}_{\mathrm{e}} := \tfrac{1}{2} \left(\hat{\mathscr{F}}_{\mathrm{e}}^{\mathsf{T}} \hat{\mathscr{F}}_{\mathrm{e}} - \hat{\mathbf{I}}^{\backslash} \right) , \tag{162}$$

$$\hat{\mathbf{E}}^{\backslash}_{\mathrm{p}} := \tfrac{1}{2} \left(\hat{\mathbf{I}}^{\backslash} - \hat{\mathscr{F}}_{\mathrm{p}}^{-\mathsf{T}} \mathscr{F}_{\mathrm{p}}^{-1} \right) . \tag{163}$$

The tensor $\hat{\mathbf{E}}^{\backslash}_{\mathrm{e}}$ serves as a Green type tensor for the elastic deformations, while $\hat{\mathbf{E}}^{\backslash}_{\mathrm{p}}$ serves as an Almansi type plastic strain tensor.

Remark 6.3: We point out that the strain measures (162) and (163) defined on the intermediate configuration involve only the mixed identity tensor $\hat{\mathbf{I}}^{\backslash}$ and not a metric tensor $\hat{\mathbf{G}}$ on the intermediate configuration as in approaches using standard definitions of push-forwards and pull-backs (e.g. [17]). This is a substantial advantage since the mixed identity tensor $\hat{\mathbf{I}}^{\backslash}$ is well defined on the entire intermediate configuration $\hat{\mathscr{B}}$. However, due to the incompatibility of the intermediate configuration the metric tensor $\hat{\mathbf{G}}$ can only be defined in an infinitesimal neighbourhood of each material particle. □

6.3 Objective time derivatives of strain measures

Using Definition 3.10 we can compute the material time derivative of the right Cauchy-Green tensor (148) and of Green's strain tensor. As noted in Section 3.3, material time derivatives of spatial tensor fields are not objective. However, we can use the concept of *Lie derivatives* introduced in Definition 3.11 to obtain objective time derivatives of spatial deformation and strain measures. From (134), making use repeatedly of (141), we obtain the following formulae for Lie derivatives of second-order tensors:

$$\mathscr{L}_{\vec{\mathbf{v}}}(\mathbf{t}^{\flat}) = \dot{\mathbf{t}}^{\flat} + \mathbf{l}^{\backslash *}\mathbf{t}^{\flat} + \mathbf{t}^{\flat}\mathbf{l}^{\backslash} , \qquad \mathscr{L}_{\vec{\mathbf{v}}}(\mathbf{t}^{\sharp}) = \dot{\mathbf{t}}^{\sharp} - \mathbf{l}^{\backslash}\mathbf{t}^{\sharp} - \mathbf{t}^{\sharp}\mathbf{l}^{\backslash *} , \tag{164), (165}$$

$$\mathscr{L}_{\vec{\mathbf{v}}}(\mathbf{t}^{\backslash}) = \dot{\mathbf{t}}^{\backslash} + \mathbf{l}^{\backslash\mathsf{T}}\mathbf{t}^{\backslash} + \mathbf{t}^{\backslash}\mathbf{l}^{\backslash} , \qquad \mathscr{L}_{\vec{\mathbf{v}}}(\mathbf{t}^{\prime}) = \dot{\mathbf{t}}^{\prime} - \mathbf{l}^{\backslash *\mathsf{T}}\mathbf{t}^{\prime} - \mathbf{t}^{\prime}\mathbf{l}^{\backslash *} . \tag{166), (167}$$

In deriving these expressions, use is made of the expression

$$\mathbf{l}^{\backslash *} = \mathscr{F}^{*-1} \dot{\mathscr{F}}^{*} \tag{168}$$

for the dual of the velocity gradient, $\mathbf{l}^{\backslash *}$, according to (141) and (65).

Remark 6.4: Note that the convective terms in the Lie derivative of mixed tensors are determined only either by the velocity gradient tensor $\mathbf{l}^{\backslash}$ (and its transpose) or by its dual $\mathbf{l}^{\backslash *}$ (and its transpose). The convective terms in Lie derivatives of covariant and contravariant tensors are determined by the velocity gradient as well by its dual. Identifying the dual of the velocity gradient with its transpose converts our formulae for the Lie derivatives of covariant and contravariant tensors to the standard formulae [1, 13]. However, the convective terms for the Lie derivatives of mixed tensors cannot be converted completely to the standard formulae. This is due to the new formulae of push-forwards and pull-backs for mixed tensors (see Table 1) which cannot be converted to the standard formulae either. □

Remark 6.5: Also note that the Lie derivative preserves duality for covariant and contravariant tensors, respectively. If $\mathbf{t}^{\flat}$ or $\mathbf{t}^{\sharp}$ are self-dual, then their Lie derivatives are also self-dual. Correspondingly symmetry is preserved for symmetric mixed tensors $\mathbf{t}^{\backslash}$ and $\mathbf{t}^{\prime}$. Of course, this quality of our formulae for Lie derivatives is closely related to the corresponding push-forwards and pull-backs represented in Table 1 (see also Remark 3.12).

Since the velocity gradient is a mixed tensor it can be symmetrized. We call the symmetric part of the velocity gradient the *rate of deformation tensor*

$$\mathbf{d}^{\backslash} := \tfrac{1}{2} \left(\mathbf{l}^{\backslash} + \mathbf{l}^{\backslash\mathsf{T}} \right) . \tag{169}$$

Straightforward application of (166) yields

$$\mathscr{L}_{\vec{\mathbf{v}}}(\mathbf{i}^{\backslash}) = 2\mathbf{d}^{\backslash} . \tag{170}$$

Remark 6.6: The counterpart to (170) in standard formulations [1] is

$$\mathscr{L}_{\vec{\mathbf{v}}}(\mathbf{g}) = 2\mathbf{d}^{\flat} . \tag{171}$$

A comparison of (170) and (171) reveals once more that in the mixed variant formulation the identity tensor $\mathbf{i}^{\backslash}$ plays a similar role as a metric tensor. □

Next, we present the material time derivative of the difference of the squared line elements in the current and reference configurations. On the current configuration we obtain the material time derivative of (157) under consideration of (147) and (166),

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathrm{d}s^2 - \mathrm{d}S^2) = 2\,\mathrm{d}\vec{\mathbf{x}} \cdot \mathscr{L}_{\vec{\mathbf{v}}}(\mathbf{e}^{\backslash})\,\mathrm{d}\vec{\mathbf{x}} . \tag{172}$$

On the other hand, material time differentiation of (152) gives under consideration of (148) and (169)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathrm{d}s^2 - \mathrm{d}S^2) = 2\,\mathrm{d}\vec{\mathbf{x}} \cdot \mathbf{d}^{\backslash}\,\mathrm{d}\vec{\mathbf{x}} . \tag{173}$$

A comparison of (172) and (173) gives

$$\mathbf{d}^{\backslash} = \mathscr{L}_{\vec{\mathbf{v}}}(\mathbf{e}^{\backslash}) , \tag{174}$$

which again is known for the covariant formulation from the literature [1].

Finally, we obtain on the reference configuration by material time differentiation of (156)

$$\frac{\mathrm{d}}{\mathrm{d}t}\,(\mathrm{d}s^2 - \mathrm{d}S^2) = 2\,\mathrm{d}\vec{\mathbf{X}} \cdot \dot{\mathbf{E}}^{\backslash}\,\mathrm{d}\vec{\mathbf{X}} . \tag{175}$$

It follows directly from (155) and application of the Lie derivative in (134) that

$$\mathbf{d}^{\backslash} = \varphi_*(\dot{\mathbf{E}}^{\backslash}) , \qquad \dot{\mathbf{E}}^{\backslash} = \varphi^*(\mathbf{d}^{\backslash}) . \tag{176}$$

Remark 6.7: In Section 6.2 and Section 6.3 we have recovered all important kinematic formulae for mixed variant strains and deformation rates which are known for covariant measures from the standard literature [1, 13]. This achievement is due to the introduction of modified equations for push-forwards of mixed second-order tensors presented in Section 3.2. □

7. Dynamics of continuous bodies

Duality being the motivation for the present mathematical formalism, this section deals with the dual of the kinematics of continua: that is dynamics.

7.1 Stresses

Duality enters into mechanics only when discussing the physical duality between "forces" and "velocities". Therefore, let us consider the body B in an arbitrary configuration $\mathscr{S}$, and let the corresponding spatial velocity field be given as $\vec{\mathbf{v}}(x)$ with $\vec{\mathbf{v}} \in T_x\mathscr{S}$. Now, we introduce the *traction one-form* $\vec{\boldsymbol{\tau}} \in T^*_x\mathscr{S}$ as a field defined on the boundary $\partial\mathscr{S}$ of $\mathscr{S}$ which governs the mechanical interactions of the body with its surroundings at the instant under consideration. Also, we introduce the *body force one-form* field $\vec{\boldsymbol{\beta}} \in T^*_x\mathscr{S}$ on $\mathscr{S}$, taken per unit volume; the body forces are assumed to include inertia forces.

Remark 7.1: It is essential to note that $\vec{\boldsymbol{\tau}}$ and $\vec{\boldsymbol{\beta}}$ are defined on the cotangent space $T^*_x\mathscr{S}$ being dual to the tangent space $T_x\mathscr{S}$ on which the velocity $\vec{\mathbf{v}}$ is defined. In conventional continuum mechanics the traction one-form and the body force one-form fields are introduced as vectors. □

The *work rate* or *power* P of the two types of external loading on the body B at the current instant is given by the scalar product

$$P_{\text{ext}} = \int_{\partial\mathscr{S}} \langle \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}} \rangle_x \,\mathrm{d}a + \int_{\mathscr{S}} \langle \vec{\boldsymbol{\beta}}, \vec{\mathbf{v}} \rangle_x \,\mathrm{d}v , \tag{177}$$

with

$$\vec{\mathbf{v}} \in T_x\mathscr{S} , \qquad \vec{\boldsymbol{\tau}}, \vec{\boldsymbol{\beta}} \in T^*_x\mathscr{S} . \tag{178}$$

Remark 7.2: As put forward in the Introduction, it is in the notion of work that the conceptual differences between inner and scalar products become manifest. The inner product is typically used to determine the length of a vector, as in (142)–(143); the scalar product is typically used to quantify work, see (177). In conventional tensor analysis, the scalar product in (177) is denoted with the same mathematical symbolism as the inner product, although evidently the physical interpretations of the two products are absolutely different.

Remark 7.3: While the geometrical interpretation of the inner product of two vectors is elementary, the geometrical interpretation of the scalar product of a vector and a one-form or covector is much less common. MISNER et al. [10] give a very neat example of the latter which relates directly to a physically motivated visualization of force vectors. Vectors in the three-dimensional space are depicted as usually by arrows with a certain length, determined by a ruler of unit length. A one-form, on the other hand, is depicted as a set of parallel planes, spaced at unit distance. The visualization of a force as a set of planes – though quite different from the traditional picture in classical continuum mechanics – becomes clear if one thinks of forces as being derived from a potential, or, as in quantum mechanics, through surfaces of de Broglie wave of equal phase. The scalar product of the vector and the one-form can then be visualized as the number of the surfaces of the one-form that is pierced through by the vector; this is illustrated schematically in Fig. 1. □

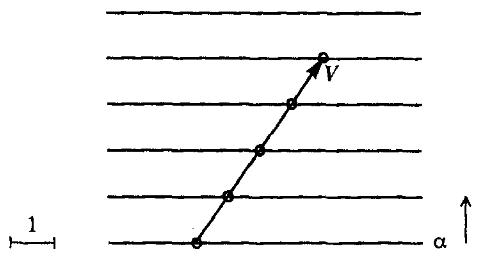

Fig. 1. Planar illustration of the visualization of the scalar product between a vector $\vec{\mathbf{v}} \in T_x\mathscr{S}$ and a one-form $\vec{\boldsymbol{\alpha}} \in T_x^*\mathscr{S}$. While $\vec{\mathbf{v}}$ is depicted as an arrow with a certain length measured by the unit stick, $\vec{\boldsymbol{\alpha}}$ is depicted as a set of parallel planes at unit distance. The scalar product $\langle \vec{\mathbf{v}}, \vec{\boldsymbol{\alpha}} \rangle_x$ is the number of planes of $\vec{\boldsymbol{\alpha}}$ that $\vec{\mathbf{v}}$ pierces through. Physically, if $\vec{\mathbf{v}}$ identifies the velocity vector of a particle at some instant, the force acting on the particle is $\vec{\boldsymbol{\alpha}}$, and the scalar product signifies the rate of work. (α on the figure corresponds to $\vec{\boldsymbol{\alpha}}$ in the text.)

In order to define the internal power consumed inside the body due to the instantaneous rate of deformation $\mathbf{d}^{\backslash}$ defined in (169), we introduce a *stress tensor* field dual to $\mathbf{d}^{\backslash}$. This stress tensor $\mathbf{s}'$ is called the *Cauchy stress tensor.* The internal power then is given by

$$P_{\text{int}} = \int_{\mathscr{S}} \langle \mathbf{s}^{/}, \mathbf{d}^{\backslash} \rangle_x \, \mathrm{d}v\,, \tag{179}$$

where

$$\mathbf{d}^{\backslash} \in T_x\mathscr{S} \otimes T_x^*\mathscr{S}, \qquad \mathbf{s}^{/} \in T_x^*\mathscr{S} \otimes T_x\mathscr{S}, \tag{180}$$

thus emphasizing duality between $\mathbf{s}^{/}$ and $\mathbf{d}^{\backslash}$.

According to the principal of virtual power, the internal and external power given through (179)–(177) have to be identical for all kinematically admissible velocity fields $\vec{\mathbf{v}}$, i.e. with $\mathbf{d}^{\backslash}$ expressed in terms of the velocity field through (169) and (139). Introducing the unit outer normals on $\partial\mathscr{S}$ in terms of a field of one-forms $\vec{\boldsymbol{\nu}} \in T_x^*\mathscr{S}$, the divergence theorem yields Cauchy's law in the form

$$\vec{\boldsymbol{\tau}} = \mathbf{s}^{/}\vec{\boldsymbol{\nu}} \quad \text{on} \quad \partial\mathscr{S}. \tag{181}$$

Remark 7.4: In this form, the Cauchy law emphasizes the mixed nature of the Cauchy stress tensor $\mathbf{s}^{/}$. On the one hand, this emerges as a consequence of the mixed nature of the rate of deformation tensor $\mathbf{d}^{\backslash}$ (see (179)); on the other hand, it follows from both $\vec{\boldsymbol{\nu}}$ and $\vec{\boldsymbol{\tau}}$ living in $T_x^*\mathscr{S}$. □

In view of the fact that $\langle \mathbf{s}', \mathbf{d}^{\backslash} \rangle_x$ measures the rate of work per unit volume in the current configuration, it is convenient to make use of the Kirchhoff stress tensor, defined by

$$\mathbf{t}^{/} = J\mathbf{s}^{/}\,, \tag{182}$$

where J is the Jacobian of the deformation gradient, $J = \det \mathscr{F}$, which is obtained most elegantly through $J = (\det \mathbf{C}^{\backslash})^{1/2} = (\det \mathbf{b}^{\backslash})^{1/2}$ (see (148), (149)).

In the foregoing, we have considered fields on the current configuration $\mathscr{S}$ of B. It is often convenient in mechanics to operate on quantities referring to the initial reference configuration $\mathscr{B}$. The external and internal power are conveniently re-phrased in terms of such quantities by pull-back operations from $\mathscr{S}$ onto $\mathscr{B}$. In particular, we have already seen the relation between the rate of deformation $\mathbf{d}^{\backslash}$ and the Green strain-rate tensor $\dot{\mathbf{E}}^{\backslash}$ in (176). Dual to this relation, we make use of the relationship

$$\mathbf{T}^{/} = \varphi^*(\mathbf{t}^{/})\,, \qquad \mathbf{t}^{/} = \varphi_*(\mathbf{T}^{/})\,, \tag{183}$$

or

$$\mathbf{T}^{/} = \mathscr{F}^{*-\mathsf{T}}\mathbf{t}^{/}\mathscr{F}^{*-1}\,, \qquad \mathbf{t}^{/} = \mathscr{F}^{*\mathsf{T}}\mathbf{T}^{/}\mathscr{F}^{*} \tag{184}$$

(see Table 1), between the Kirchhoff stress tensor $\mathbf{t}^{/}$ and the second Piola-Kirchhoff stress tensor $\mathbf{T}^{/}$. Making use of the well-known geometric identity $\mathrm{d}v = J\,\mathrm{d}V$, the internal power (179) can then be written entirely in terms of quantities on $T_X\mathscr{B}$ and its dual as

$$P_{\rm int} = \int_{\mathscr{B}} \langle \mathbf{T}^{/}, \dot{\mathbf{E}}^{\backslash} \rangle_X \, dV, \tag{185}$$

where

$$\dot{\mathbf{E}}^{\backslash} \in T_X\mathscr{B} \otimes T^*_X\mathscr{B}, \qquad \mathbf{T}^{/} \in T^*_X\mathscr{B} \otimes T_X\mathscr{B}. \tag{186}$$

Remark 7.5: It is instructive to compare the transformation formulae (184) with those in the traditional notation, where the difference between, for instance, $\mathscr{F}^*$ and $\mathscr{F}^{\mathsf{T}}$ has faded. In particular, notice that while the transformations between strain rates are governed by $\mathscr{F}$, the transformations between stresses are governed by its dual, $\mathscr{F}^*$.

7.2 Objective time derivatives of stresses

Just like for the case of strain measures, discussed in Section 6.3, objective time derivatives of the spatial stress measures of Cauchy, $\mathbf{s}^{/}$, and of Kirchhoff, $\mathbf{t}^{/}$, will be formulated by application of the Lie derivative. It is clear with reference to (167), that the Lie derivative of the Kirchhoff stress is

$$\mathscr{L}_{\tilde{\mathbf{v}}}(\mathbf{t}^{/}) = \dot{\mathbf{t}}^{/} - \mathbf{l}^{\backslash *\mathsf{T}}\mathbf{t}^{/} - \mathbf{t}^{/}\mathbf{l}^{\backslash *} \tag{187}$$

(see also (184)). The Lie derivative of Cauchy stress is then found from (182) by making use of the kinematic relationship $\dot{J}/J = \operatorname{tr} \mathbf{d}^{\backslash}$:

$$\mathscr{L}_{\tilde{\mathbf{v}}}(\mathbf{s}^{/}) = \dot{\mathbf{s}}^{/} - \mathbf{l}^{\backslash *\mathsf{T}}\mathbf{s}^{/} - \mathbf{s}^{/}\mathbf{l}^{\backslash *} + \mathbf{s}^{/} \operatorname{tr} \mathbf{d}^{\backslash *}. \tag{188}$$

In writing the last term, we have also made use of the general property $\operatorname{tr} \mathbf{d}^{\backslash *} = \operatorname{tr} \mathbf{d}^{\backslash}$, which is immediately clear from (60).

Remark 7.6: It is interesting to note that in the present formulation, the Lie stress rates depend exclusively on the dual velocity gradient $\mathbf{l}^{\backslash *}$. Of course, it is a direct consequence of the mixed nature of the stress tensors and the associated push-forwards and pull-backs. But, the appearance of $\mathbf{l}^{\backslash *}$ as a tensor in $T^*_x\mathscr{S} \otimes T_x\mathscr{S}$ can also be seen directly to ensure that the stress tensors and their respective Lie derivative live in that same space $T^*_x\mathscr{S} \otimes T_x\mathscr{S}$. These forms therefore provide an important distinction from the expressions in the more standard notation. If indeed one drops the distinction between inner and scalar products, the Lie derivatives reduce to the so-called *Truesdell stress rates.*

After decomposition of $\mathbf{l}^{\backslash *}$ into its symmetric part, $\mathbf{d}^{\backslash *}$, and its skewsymmetric part, $\mathbf{w}^{\backslash *}$, i.e.

$$\mathbf{d}^{\backslash *} = \tfrac{1}{2}(\mathbf{l}^{\backslash *} + \mathbf{l}^{\backslash *\mathsf{T}}), \qquad \mathbf{w}^{\backslash *} = \tfrac{1}{2}(\mathbf{l}^{\backslash *} - \mathbf{l}^{\backslash *\mathsf{T}}), \tag{189}$$

the expression (188) can be rewritten as

$$\mathscr{L}_{\tilde{\mathbf{v}}}(\mathbf{s}^{/}) = \overset{\nabla}{\mathbf{s}}{}^{/} - \mathbf{d}^{\backslash *\mathsf{T}}\mathbf{s}^{/} - \mathbf{s}^{/}\mathbf{d}^{\backslash *} + \mathbf{s}^{/} \operatorname{tr} \mathbf{d}^{\backslash *}, \tag{190}$$

where $\overset{\nabla}{\mathbf{s}}{}^{/}$ is the *Jaumann derivative* of $\mathbf{s}^{/}$,

$$\overset{\nabla}{\mathbf{s}}{}^{/} = \dot{\mathbf{s}}^{/} + \mathbf{w}^{\backslash *}\mathbf{s}^{/} - \mathbf{s}^{/}\mathbf{w}^{\backslash *}. \tag{191}$$

Remark 7.7: Due to the distinction maintained here between inner and scalar products, the Jaumann derivation appears here in a rather unusual form. In particular, note the signs of the convective terms that appear in terms of the dual of the spin tensor, $\mathbf{w}^{\backslash *} \in T^*_x\mathscr{S} \otimes T_x\mathscr{S}$. However, it should be realized that we have the relationship $\mathbf{w}^{\backslash *} = -\mathbf{G}^\flat \mathbf{w}^{\backslash} \mathbf{G}^\sharp$, by virtue of $\mathbf{w}^{\backslash *} = \mathbf{G}^\flat \mathbf{w}^{\backslash \mathsf{T}} \mathbf{G}^\sharp$ (see (87)) and skewsymmetry of $\mathbf{w}^{\backslash}$. Thus, the formulation (191) is fully consistent with the traditional form. □

8. Application to constitutive equations

The previous section has focussed on the dual vectors and tensors that are central in continuum mechanics: displacements/velocities and strains/strain-rates on the one hand, forces and stress tensors on the other. The link between the two sets of quantities is furnished by constitutive relations, as mappings between the mutually dual spaces of strain (rate) and stress (rate). When referred to the reference configuration, the strain space is the tensor space $T_X\mathscr{B} \otimes T^*_X\mathscr{B}$ while the stress space is a tensor space $T^*_X\mathscr{B} \otimes T_X\mathscr{B}$ (see (186)). In terms of quantities in the current configuration, constitutive equations provide a mapping between the strain space $T_x\mathscr{S} \otimes T^*_x\mathscr{S}$ and the stress space $T^*_x\mathscr{S} \otimes T_x\mathscr{S}$ (see (180)). In this section, we emphasize a number of general features of such constitutive relations that are brought about clearly by the present mathematical formalism.

8.1 Potentials

For clarity of the arguments, let us consider constitutive equations for a linear solid that are determined through an energy density function W. It is well-know that it is convenient to define the elastic energy under isothermal conditions to be a function of the Green strain tensor $\mathbf{E}^{\backslash} \in T_X\mathscr{B} \otimes T^*_X\mathscr{B}$ in order that W is properly invariant under rigid body rotations superimposed on the current configuration. Thus, the elastic energy density function,

$$W = W(\mathbf{E}^{\backslash}) \in \mathbb{R}, \tag{192}$$

leads to the constitutive equation for a hyperelastic solid,

$$\mathbf{T}^{/} = \partial W / \partial \mathbf{E}^{\backslash}, \tag{193}$$

by adopting it as a *potential*. What is interesting to note here, is that the gradient of W as defined formally in Definition 2.20 ensures that $\partial W/\partial \mathbf{E}^{\backslash}$ belongs to the stress space $T_X^*\mathscr{B} \otimes T_X\mathscr{B}$.

Remark 8.1: This observation has more far-reaching consequences: if $\mathbf{E}^{\backslash}$ is restricted to live in a subspace of $T_X\mathscr{B} \otimes T_X^*\mathscr{B}$, the potential formulation assures that $\mathbf{T}^{/}$ can only move in the dual subspace. For instance, $\mathbf{T}^{/}$ immediately inherits symmetry from $\mathbf{E}^{\backslash}$. Furthermore, if for an incompressible solid $\mathbf{E}^{\backslash}$ is constrained to live in the deviatoric subspace of $T_X\mathscr{B} \otimes T_X^*\mathscr{B}$, the stress tensor according to the potential formulation (193) belongs to the deviatoric subspace of $T_X^*\mathscr{B} \otimes T_X\mathscr{B}$. □

The same constitutive behaviour can be formulated in terms of quantities in the current configuration by employing the push-forwards in (155) and (183). Then,

$$\mathbf{t}^{/} = \frac{\partial W}{\partial \mathbf{e}^{\backslash}}, \qquad \mathbf{t}^{/} = \mathscr{F}^{*\mathrm{T}} \frac{\partial W}{\partial \mathbf{E}^{\backslash}} \mathscr{F}^{*}, \tag{194, 195}$$

with the elastic potential now formulated as

$$W = W(\varphi^*(\mathbf{e}^{\backslash})) = W(\mathscr{F}^{\mathrm{T}}\mathbf{e}^{\backslash}\mathscr{F}). \tag{196}$$

Here, again note the duality of the transformations between $T_X\mathscr{B}$ and $T_x\mathscr{S}$ involved.

Potentials that are in a sense dual to the potentials discussed above, are used to formulate constitutive equations for viscous solids. Then, a dissipative or viscous potential Φ, for instance, defined as

$$\Phi = \Phi(\varphi^*(\mathbf{t}^{/})), \tag{197}$$

determines the dual strain rate through

$$\mathbf{d}^{\backslash} = \partial \Phi / \partial \mathbf{t}^{/}. \tag{198}$$

Note that the potential Φ is properly invariant to superimposed rigid body rotations since it is defined as a function of the pull-back of the stress tensor $\mathbf{t}^{/}$.

8.2 Linear constitutive equations

The potentials discussed in the previous subsection can be used quite generally for nonlinear materials. By specializing the functional forms of the potential functions to quadratic forms of their argument, linear constitutive equations are obtained. As an example, let us consider a quadratic elastic potential $W(\mathbf{E}^{\backslash})$. This is conveniently expressed through a scalar product:

$$W(\mathbf{E}^{\backslash}) = \tfrac{1}{2}\,\langle \mathbf{E}^{\backslash}, \mathbf{C}^{//}\mathbf{E}^{\backslash} \rangle_X, \tag{199}$$

where $\mathbf{C}^{//}$ is the fourth-order tensor of elastic moduli, being an element of $T_X^*\mathscr{B} \otimes T_X\mathscr{B} \otimes T_X^*\mathscr{B} \otimes T_X\mathscr{B}$. Rewriting this expression in terms of the dual tensor $\mathbf{C}^{//*}$,

$$W(\mathbf{E}^{\backslash}) = \tfrac{1}{2}\,\langle \mathbf{C}^{//*}\mathbf{E}^{\backslash}, \mathbf{E}^{\backslash} \rangle_X, \tag{200}$$

it is immediately seen that $\mathbf{C}^{//*} = \mathbf{C}^{//}$ so that $\mathbf{C}^{//}$ is a *self-dual* tensor (see (69)).

Remark 8.2: Notice the doubly-mixed nature of the moduli tensor $\mathbf{C}^{//}$. This is a consequence of the fact that in this approach, strain and stress tensors emerge as mixed tensors. With dual bases $\vec{\mathbf{E}}_A \in T_X\mathscr{B}$ and $\vec{\boldsymbol{\varepsilon}}_0{}^A \in T_X^*\mathscr{B}$, respectively, the components of $\mathbf{C}^{//}$ are written $\mathrm{C}_A{}^B{}_C{}^D$. □

Substitution of this potential into (193) yields the following linear constitutive relation,

$$\mathbf{T}^{/} = \mathbf{C}^{//}\mathbf{E}^{\backslash}. \tag{201}$$

When $\mathbf{C}^{//}$ is taken to be a constant tensor, this relationship embodies Hooke's law with respect to the undeformed configuration.

Remark 8.3: The expression (201) emphasizes the essential physical difference between symmetry of a tensor and self-duality of a tensor. Both $\mathbf{T}^{/}$ and $\mathbf{E}^{\backslash}$ are symmetric tensors; but, $\mathbf{C}^{//}$ is self-dual. Self-duality of a tensor like $\mathbf{C}^{//}$ is a consequence of, for instance, the existence of a potential (see (199)). Symmetry has an entirely different background. For instance, symmetry of $\mathbf{E}^{\backslash}$ physically stems from the fact that $\mathbf{E}^{\backslash}$ and $\mathbf{E}^{\backslash\mathrm{T}}$ measure the same change of length (see (156)). Obviously, the moduli tensor also inherits the symmetry of $\mathbf{E}^{\backslash}$. In terms of components $\mathrm{C}_A{}^B{}_C{}^D$, this is expressed through the symmetries ${}_A{}^B \leftrightarrow {}^B{}_A$ and ${}_C{}^D \leftrightarrow {}^D{}_C$ (cf. (49) and (95)). The self-duality of course, is expressed through the symmetry ${}_A{}^B{}_C{}^D \leftrightarrow {}_C{}^D{}_A{}^B$. □

9. Conclusions

The conclusions for Part 1 are given in Section 5. Therefore, we concentrate here on the results of Part 2. In Section 6 we apply our mathematical results to the kinematics of motions of continuous bodies. We present a consistently mixed variant formulation of deformation tensors, strain tensors, and rate of deformation tensors, where our new concepts of push-forwards and pull-backs, respectively, play a central role. All results known for the more conventional formulation based on covariant tensors are recovered for the mixed variant formulation. As an interesting example we consider kinematics of finite deformation plasticity based on the multiplicative decomposition of the deformation gradient. We present the additive decomposition of the strain tensor on the intermediate configuration. Contrary to the covariant formulation, in our approach the mixed identity tensor enters instead of the metric tensor. The latter, due to the incompatibility of the intermediate configuration, can only be defined in an infinitesimal neighbourhood of a particle, whereas the mixed identity tensor is well-defined globally on the intermediate configuration.

In Section 7 we consider stresses and stress rates. We first compute the external and internal power. It turns out that the stress "vector" is a one-form which lives in the dual space $T_x^*\mathscr{S}$ to the space $T_x\mathscr{S}$ of the velocity. Correspondingly the Cauchy stress $\mathbf{s}^{/} \in T_x^*\mathscr{S} \otimes T_x\mathscr{S}$ lives in the dual of the rate of deformation space $T_x\mathscr{S} \otimes T_x^*\mathscr{S}$. As objective time rates we compute the Lie derivative of the Cauchy stress and the Jaumann derivative. Both objective derivatives are expressed entirely in terms of the dual velocity gradient $\mathbf{l}^{\backslash *}$ and its antisymmetric part, respectively.

Finally, in Section 8 we consider constitutive equations which can be derived from potentials. Again, duality between stresses and strains emerges naturally in our formulation. This is demonstrated in an exemplaric manner for a hyperelastic and a viscous solid. If the potential for the hyperelastic solid is a quadratic form in the strain tensor, then Hooke's law of linear elasticity is recovered. The fourth-order elasticity tensor is self dual (as a consequence of the existence of a potential) and symmetric, since strain tensors and their duals measure the same length.

For **References** cf. Part 1, ZAMM **76** (1996) 8, 447–462.

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